Dirac free Particle solution or plane wave solution → (1)

The wave function ψ has 4 components and the Dirac equation is a set of 4 First order linear partial differential equation, the wave function will have the form.

$$\psi_j(r,t) = u_j e^{i(k.r-\omega t)} \quad \text{— (1)}$$

$\psi(r,t)$ are the eigen function of energy and momentum. with eigen values.

$E = \hbar\omega$ and $p = \hbar k$ respectively

$$\left.\begin{aligned}
(E-mc^2)u_1 - cP_z u_3 - c(P_x - iP_y)u_4 &= 0\\
(E-mc^2)u_2 - c(P_x+iP_y)u_3 + cP_z u_4 &= 0\\
(E+mc^2)u_3 - cP_z u_1 - c(P_x-iP_y)u_2 &= 0\\
(E+mc^2)u_4 - c(P_x+iP_y)u_1 + cP_z u_2 &= 0
\end{aligned}\right\} \quad \text{— (2)}$$

These equation being homogeneous will have non-trivial solution only if the determinent of the Coeff. is zero.

$$\begin{vmatrix}
E-mc^2 & 0 & -cP_z & -c(P_x-iP_y) \\
0 & E-mc^2 & -c(P_x+iP_y) & cP_z \\
-cP_z & -c(P_x-iP_y) & (E+mc^2) & 0 \\
-c(P_x+iP_y) & cP_z & 0 & E+mc^2
\end{vmatrix} = 0 \quad \text{— (3)}$$

or $(E^2 - m^2c^4 - p^2c^2)^2 = 0$

This is momentum energy relation for free particle

The determinant 3 is of rank 2. To show this we replace it by Δ ②

$$\Delta = (E - mc^2)\begin{vmatrix} E - mc^2 & -c(P_x + iP_y) & cP_z \\ -c(P_x - iP_y) & E + mc^2 & 0 \\ cP_z & 0 & (E+mc^2) \end{vmatrix}$$

$$-cP_z \begin{vmatrix} 0 & (E - mc^2) & cP_z \\ -cP_z & -c(P_x - iP_y) & 0 \\ -c(P_x + iP_y) & 0 & (E + mc^2) \end{vmatrix}$$

$$-c(P_x - iP_y)\begin{vmatrix} 0 & (E - mc^2) & c(P_z + iP_y) \\ -cP_z & -c(P_x - iP_y) & E + mc^2 \\ -c(P_x + iP_y) & cP_z & 0 \end{vmatrix}$$

After solving we obtain

$$\Delta = (E^2 - m^2c^4 - c^2p^2)(E^2 - m^2c^4 - c^2p^2) = 0$$

$$E = \pm (p^2c^2 + m^2c^4)^{1/2}$$

Hence we can assign numerical values to two of u's. We choose term conveniently and obtain linearly dependent solution

$$E_+ = +(p^2c^2 + m^2c^4)^{1/2}$$

$$u_1 = 1,\ u_2 = 0,\ u_3 = \frac{cP_z}{E_+ + mc^2},\ u_4 = \frac{c(P_x + iP_y)}{E_+ + mc^2} \quad (3)$$

$$u_1 = 0,\ u_2 = 1,\ u_3 = \frac{c(P_x - iP_y)}{E_+ + mc^2},\ u_4 = \frac{-cP_z}{E_+ + mc^2} \quad (4)$$

There are two linearly independent solution

Similarly we choose the negative square root

$$E_- = -(c^2p^2 + m^2c^4)^{1/2}$$

We obtain new solution

$$u_1 = \frac{cP_z}{E_- - mc^2},\ u_2 = \frac{c(P_x - iP_y)}{E_- - mc^2},\ u_3 = 1,\ u_4 = 0$$

$$u_1 = \frac{c(P_x - iP_y)}{E_- - mc^2},\ u_2 = \frac{-cP_z}{E_- - mc^2},\ u_3 = 0,\ u_4 = 1$$

For each solution we can be normalised.

$$\psi^*\psi = 1$$

$$u_1^*u_1 + u_2^*u_2 + u_3^*u_3 + u_4^*u_4 = 1$$

$$N^2\left[1 + 0 + \frac{c^2P_z^2}{(E_+ + mc^2)^2} + \frac{c^2(P_x^2 + P_y^2)}{(E_+ + mc^2)^2}\right] = 1$$

$$N^2 = \frac{1}{1 + \frac{c^2(P_x^2 + P_y^2 + P_z^2)}{(E_+ + mc^2)^2}} = \frac{1}{1 + \frac{p^2c^2}{(E_+ + mc^2)^2}}$$

$$N = \left[1 + \frac{c^2p^2}{(E_+ + mc^2)^2}\right]^{1/2}$$